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ON THE MITCHELL PROBLEM OF THE MOTION OF A LUBRICANT IN A LAYER BOUNDED BY A MOVING PLANE AND A FIXED PLATE OF FINITE SIZES*

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The Reynolds equation which appears in the hydrodynamic theory of lubrication is applied to the case of the flow of a lubricant between a plane and an inclined plate, and is solved with help of the special functions for a rectangular as well as segmented form of the plate.

1. An unbounded plane moves longitudinally with velocity U in the direction of the x axis. We direct the y axis towards the liquid. Let h be the thickness of the layer, depending only on the coordinate x, $q = h_2/h_1 > 1$ the ratio of the thicknesses of the layer at the plate edges along the x axis, a the distance between these edges and $2l$ the width of the plate in the direction of the z axis.

Using the well-known approximate Reynolds equation, we arrive at the following boundary value problem for the pressure:

$$\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right)+h^3\frac{\partial^2 p}{\partial z^2}=6\mu U\frac{\partial h}{\partial x} \tag{1.1}$$

$$\begin{gathered}-l<z<l,\ x=0,\ p=p_a;\ x=a,\ p=p_a\\ 0<x<a,\ z=\pm l,\ p=p_a\end{gathered} \tag{1.2}$$

Since h depends only on x, it follows that a particular solution of Eq.(1.1) can be taken in the form

$$p_0=\chi_0(x)=6\mu U\int h^{-2}dx+C_1\int h^{-3}dx+C_2 \tag{1.3}$$

We shall construct the solution of the corresponding homogeneous equation in the form $p_n=\operatorname{ch}(nz)\chi_n(x)$. In this case we obtain the following equation for χ_n:

$$\frac{d}{dx}\left(h^3\frac{d\chi_n}{dx}\right)+n^2h^3\chi_n=0 \tag{1.4}$$

whose complete solution will consist of two independent solutions $\chi_n^{(1)}$ and $\chi_n^{(2)}$, so that $\chi_n=A_n\chi_n^{(1)}+B_n\chi_n^{(2)}$.

We can show in the usual manner that the functions χ_n are orthogonal

$$\int_0^a \chi_m\chi_n h^3dx=0,\quad m\neq n \tag{1.5}$$

when the following conditions hold:

$$A_n\chi_n^{(1)}(0)+B_n\chi_n^{(2)}(0)=0,\quad A_n\chi_n^{(1)}(a)+B_n\chi_n^{(2)}(a)=0 \tag{1.6}$$

Combining the particular solution (1.3) with the set of solutions χ_n, we obtain the general solution of Eq.(1.1) in the form

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$$p = \chi_0(x) + \sum_n \operatorname{ch}(nz)\,[A_n\chi_n^{(1)} + B_n\chi_n^{(2)}] \tag{1.7}$$

Satisfying the first two conditions of (1.2), we obtain

$$C_1 = -6\mu U \int_0^a h^{-2}dx \left[\int_0^a h^{-3}dx\right]^{-1}, \quad C_2 = p_a$$

and (1.6) will yield the following equations for the eigenvalues:

$$\chi_n^{(1)}(0)\,\chi_n^{(2)}(a) - \chi_n^{(1)}(a)\,\chi_n^{(2)}(0) = 0 \tag{1.8}$$

while the functions χ_n will be written in the form

$$\chi_n = A_n D_n(x)$$

$$D_n(x) = \chi_n{}^2(x)\,\chi_n^{(1)}(a)\,[\chi_n^{(2)}(a) - \chi_n^{(1)}(x)]^{-1}$$

Substituting this expression into (1.7) and satisfying the last condition of (1.2), we obtain

$$6\mu U \int_0^x h^{-2}dx + C_1 \int_0^x h^{-3}dx = \sum_n A_n \operatorname{ch}(nl)\, D_n(x) \tag{1.9}$$

In order to obtain the coefficients A_n, from Eq.(1.9), we must utilize the property of orthogonality of (1.5). Multiplying both sides of (1.9) by $h^3D_n(x)$ and integrating from $x = 0$ to $x = a$, we obtain

$$A_n = 6\mu U\left[\operatorname{ch}(nl)\int_0^a D_n{}^2(x)\,h^3dx\right]^{-1}\int_0^a\left[D_nh^3\int_0^x h^{-2}(x_1)\,dx_1\right]dx + C_1\int_0^a\left[D_nh^3\int_0^x h^{-3}(x_1)\,dx_1\right]dx$$

In the special case of an inclined flat plate, we have

$$h = h_2 - mx, \quad m = h_1a^{-1}(q-1)$$

and Eq.(1.4) will take the form

$$\frac{d^2\chi_n}{dh^2} + 3h^{-1}\frac{d\chi_n}{dh} + n^2m^{-2}\chi_n = 0 \tag{1.10}$$

We can represent the solution of Eq.(1.10) in terms of Bessel functions of first and second kind /1/

$$\chi_n(h) = h^{-1}\,[A_nJ_1(nhm^{-1}) + B_nY_1(nhm^{-1})]$$

and Eq.(1.5) for the eigenvalues will take the form

$$J_2(\zeta)\,Y_1(q\zeta) - J_1(q\zeta)\,Y_1(\zeta) = 0, \quad \zeta = na(q-1)^{-1} \tag{1.11}$$

We have the following relations for the constants A_n:

$$\sum_n A_nh^{-1}Y_1^{-1}(\zeta_n)\operatorname{ch}[la^{-1}(q-1)\zeta_n]\,[Y_1(\zeta_n)J_1(hh_1^{-1}\zeta_n) - J_1(\zeta_n)Y_1(hh_1^{-1}\zeta_n)] =$$
$$6\mu aU(q-1)^{-1}(h_1^{-1}h^{-1} - h_1^{-2})\,[1 - q(1+q)^{-1}h_1(h^{-1} - h_2^{-1})]$$

where ζ_n are the roots of Eq.(1.11). We know /2/ that all these roots are real and simple, and /2/ gives some of them in a tabular form.

In the solution of the Mitchell problem constructed above we have used the hyperbolic cosine function of the coordinate z, while in the solution of Mitchell himself /3/ the coordinate z appears under the sign of the trigonometric sine and instead of the Bessel functions of first kind, functions of the third kind are used. We have used above the classical method of eigenfunctions, while Mitchell himself used a particular form of a trigonometric series.

2. In constructing the solution of the Mitchell problem for a segmented plate, we shall assume that an unbounded plane rotates about the y axis with an angular velocity of ω. The Reynolds equation for the pressure in cylindrical coordinates and the boundary conditions, have the form

$$\frac{\partial}{\partial r}\left(rh^3\frac{\partial p}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \alpha}\left(h^3\frac{\partial p}{\partial \alpha}\right) = -6\mu\omega r\frac{\partial h}{\partial \alpha} \tag{2.1}$$

$$r_1 < r < r_2, \ \alpha = 0, \ p = p_a; \ \alpha = \alpha_0, \ p = p_a, \tag{2.2}$$

$$0 < \alpha < \alpha_0, \ r = r_1, \ p = p_a; \ r = r_2, \ p = p_a$$

If the segmented plate is inclined towards the moving plane, we can put $h = h_2 - m\alpha, \ m = h_1 (q - 1) \alpha_0^{-1}$ and Eqs.(2.1) will take the form

$$r \left[\frac{\partial}{\partial r} \left(r \ \frac{\partial p}{\partial r} \right) + m^2 \left(\frac{\partial^2 p}{\partial h^2} + 3h^{-1} \frac{\partial p}{\partial h} \right) = -6\mu m\omega r^2 h^{-3} \right. \tag{2.3}$$

Taking the solution of the homogeneous equation in the form

$$p_n = (A_n r^n + B r^{-n}) \varphi_n (h) \tag{2.4}$$

we obtain for φ_n an equation of the form (1.10). Because of that, Eq.(1.11) will be retained for the eigenvalues and the degree n of the terms in (2.4) will be connected with its roots ζ_n by the relation $a^{-1} (q - 1) \zeta_n = n$. If we take the particular solution of the complete Eq. (2.3) in the form $p_0 = -6\mu m\omega r^2 f_0 (h)$, we obtain the following equation for f_0:

$$\frac{d^2 f_0}{dh^2} + 3h^{-1} \frac{df_0}{dh} + 4m^{-2} f_0 = m^{-2} h^{-3}$$

Solving this equation by varying the arbitrary constants we obtain

$$hf_0 (h) = J_1 (2hm^{-1}) \left[A_0 + \pi 2^{-1} m^{-2} \int_{h_2}^{h} J_1 (2hm^{-1}) \, h^{-1} dh \right] + Y_1 (2hm^{-1}) \left[B_0 - \pi 2^{-1} m^{-2} \times \int_{h_2}^{h} Y_1 (2hm^{-1}) \, h^{-1} dh \right]$$

The constants A_0 and B_0 can be found from the conditions (2.2) under the following constraint: $\zeta_n (q - 1) a^{-1} \neq 2$.

The complete solution of Eq.(2.3) will be

$$p = p_a - 6\mu m\omega r^2 f_0 (h) + \sum_n (A_n r^n + B_n r^{-n}) \, h^{-1} Y_1 (nh_2 m^{-1}) \, [J_1 (nhm^{-1}) \, Y_1 (nh_2 m^{-1}) - Y_1 (nhm^{-1}) \, J_1 (nh_2 m^{-1})]$$

and we shall have the following relations for the coefficients A_n and B_n:

$$6\mu m\omega r_i^2 f_0 (h) = \sum_n (A_n r_i^n + B_n r_i^{-n}) \, h^{-1} Y_1 (q\zeta_n) \, [J_1 (h\zeta_n h_1^{-1}) \, Y_1 (q\zeta_n) - Y_1 (h\zeta_n h_1^{-1}) \, J_1 (q\zeta_n)], \quad i = 1, 2$$

We can solve in the same manner the Mitchell problem for a curved segmented plate, e.g., when we write

$$h = h_2 \exp (-\beta\alpha), \ \beta = \alpha_0^{-1} \ln q$$

In this case the following function appears in place of the Bessel function:

$$\exp ({}^3/_2 \beta\alpha) \, [C_n \cos (\lambda_n \alpha) + D_n \sin (\lambda_n \alpha)], \ \lambda_n^2 = n^2 - {}^9/_4 \beta^2$$

and the form of the particular solution of the inhomogeneous equation for the pressure will also change in an appropriate manner.

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